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An Explanation of the Turbulent Round-Jet/Plane-Jet Anomaly

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Nomenclature

$C_\mu, C_{\epsilon 1}, C_{\epsilon 2}, C_{\epsilon 3}$	= turbulence model constants
℄	= centerline value
D/Dt	= $(\partial/\partial t) + U_i(\partial/\partial x_i)$, rate of change along a mean streamline
k	= kinetic energy of turbulence
P	= production rate of k
r	= radius in axisymmetric coordinates
s_{ij}	= rate-of-strain tensor
U_i	= mean-velocity vector
$\overline{u_i u_j}$	= Reynolds-stress tensor
V	= radial velocity (axisymmetric flow)
x_i	= position vector
$y_{1/2}$	= jet half-width
δ_{ij}	= Kronecker delta
ϵ	= rate of dissipation of k
μ_{eff}	= effective viscosity
$\sigma_k, \sigma_\epsilon$	= turbulence model constants
χ	= vortex-stretching invariant
ω_{ij}	= rotation tensor

Introduction

THE use of turbulence models to calculate the properties of free turbulent flows is now quite common.[1-3] Models that solve transport equations for two or more quantities have the potential advantage of generality since they require no direct empirical input such as a mixing-length specification. Their only empirical input is five or six constants which, for generality, are supposed to take the same values in all flows. However, this generality is found not to exist. Using the values of the constants appropriate to boundary-layer flows, the velocity field in a two-dimensional plane jet is calculated quite accurately, but large errors occur for axisymmetric jets. Specifically, the spreading rate of the round jet is overestimated by about 40%. Experimental data indicate that the round jet spreads about 15% less rapidly than the plane jet, while its calculated spreading rate is 15% greater.

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The discrepancy between calculated and measured values is found with both mean-flow and Reynolds-stress closures. Here, the simpler mean-flow closure is considered, and attention is focused on the k-ϵ model. For constant (unit) density flows, this model determines the Reynolds stresses through the isotropic viscosity hypothesis,

$$\overline{u_i u_j} = \frac{2}{3} k\delta_{ij} - \mu_{\text{eff}}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \tag{1}$$

where the effective viscosity is given by

$$\mu_{\text{eff}} \equiv C_\mu k^2/\epsilon \tag{2}$$

k, the kinetic energy of turbulence $(\frac{1}{2}\overline{u_i u_i})$, and ϵ, the rate of dissipation of k, are determined from transport equations:

$$\frac{Dk}{Dt} = \frac{\partial}{\partial x_i}\frac{\mu_{\text{eff}}}{\sigma_k}\frac{\partial k}{\partial x_i} + P - \epsilon \tag{3}$$

$$\frac{D\epsilon}{Dt} = \frac{\partial}{\partial x_i}\frac{\mu_{\text{eff}}}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial x_i} + \frac{\epsilon}{k}(C_{\epsilon 1}P - C_{\epsilon 2}\epsilon) \tag{4}$$

P is the rate of production of kinetic energy:

$$P = -\overline{u_i u_j}\frac{\partial U_i}{\partial x_j} \tag{5}$$

Commonly used values for the constants C_μ, $C_{\epsilon 1}$, $C_{\epsilon 2}$, σ_k and σ_ϵ are 0.09, 1.45, 1.90, 1.0, and 1.3, respectively. With these values, wall boundary layers and the plane jet are well represented.

Previous Work

In order to obtain accurate calculations of round jets, some modification to the model is required. Changing $C_{\epsilon 1}$ to 1.6 produces the desired effect but, in so doing, any notion of generality has to be abandoned. Three attempts have been made previously to introduce a modification while retaining some semblance of generality. All of these make reference to centerline values (℄) and involve modifying either $C_{\epsilon 1}$ or $C_{\epsilon 2}$:

Launder et al.[1]

$$C_{\epsilon 2} = 1.92 - .0667\left\{\frac{y_{1/2}}{2U_℄}\left(\left|\frac{dU_℄}{dx}\right| - \frac{dU_℄}{dx}\right)\right\}^{0.2} \tag{6}$$

McGuirk and Rodi[4]

$$C_{\epsilon 1} = 1.14 - 5.31\frac{y_{1/2}}{U_℄}\frac{dU_℄}{dx} \tag{7}$$

Morse[5]

$$C_{\epsilon 1} = 1.4 - 3.4\left(\frac{k}{\epsilon}\frac{\partial U}{\partial x}\right)^3_℄ \tag{8}$$

$y_{1/2}$ is the distance from the centerline to the location where the velocity is half the centerline velocity. (Launder et al. also made a modification to C_μ but this has no great effect for jets.) The different modifications to $C_{\epsilon 1}$ and $C_{\epsilon 2}$ produce the same desired effect for self-similar jets but differ slightly in the developing region; here, Morse's proposal fares best.

No convincing physical explanation is provided to justify these modifications and, in addition, there are two separate objections to them. These objections, though separate, both stem from the use of centerline values. First, the modifications imply "action at a distance"; a change in the centerline conditions is supposed immediately to affect the extremities of the jet. This is physically implausible. Second, the overall generality of the model has to be forfeited in order to provide accuracy for only two flows. The appearance of centerline values and $y_{1/2}$ in the formulas restricts their use to flows in which these quantities can be defined unambiguously. Consequently, for opposed jets or merging jets the model can only be applied in an ad hoc manner and, for a general flow, it cannot be used at all.

Analysis

Here, an explanation of the phenomenon is suggested and, as a result, a modification to the model is proposed. This modification overcomes the two objections mentioned earlier by making reference only to local events and doing so in a general, coordinate- and flow-independent manner. Like the previous suggestions, it is concerned with the dissipation process which, roughly, can be thought of as follows. Molecular viscosity causes dissipation to occur on the smallest (Kolmogoroff) scales of turbulence. Turbulence, produced on the far larger integral scales, is reduced in size by the self-action of the turbulence and by the action of the mean flowfield. This scale-reduction process becomes more and more rapid as the scale decreases until, finally, the Kolmogoroff scale is reached and the turbulence is dissipated. Consequently, the rate of dissipation is governed by the rate of scale reduction and, further, the rate-controlling process is the breaking up of the large, energy-containing motions. The dissipation rate, ϵ, can therefore more meaningfully be regarded as the rate of energy transfer from the energy-containing scales.

The suggestion made here is that the stretching of turbulent vortex tubes by the mean flow has a significant influence on the process of scale reduction. If, at some point and time, the mean motion is undergoing a positive normal strain in the direction of the turbulent vorticity vector, then this strain will tend to stretch the vortex. In order to conserve angular momentum the vortex will increase in frequency and decrease in width. On the other hand, if the strain is negative, the vortex will slow down and expand. Thus, if, on average, the turbulent vorticity vector is subjected to a positive mean normal strain, then the vortex stretching enhances the rate of scale reduction. On the small scales, the turbulent fluctuations are independent of the mean flowfield, and so the vorticity vector has no preferred direction. However, the vorticity of the large turbulent motions tends to be aligned with the vorticity in the mean flow. Thus, the mean straining of turbulent vorticity is strongly correlated with the mean straining of mean vorticity. Therefore, in flow regions where the mean vorticity is being stretched, so also is the turbulent vorticity, leading to greater scale reduction, greater dissipation, less kinetic energy, and so to a lower effective viscosity.

In two-dimensional flows, no vortex stretching can take place since the mean vorticity vector is normal to the plane of the flow; for an axisymmetric jet, however, as the jet spreads, rings of vorticity are stretched. According to this argument, this causes the effective viscosity and hence the spreading rate to be lower in the round jet than in the plane jet. This is the experimental observation.

The nondimensional measure of vortex stretching is the invariant $\chi \equiv \omega_{ij}\,\omega_{jk}\,s_{ki}$, where

$$s_{ij} \equiv \frac{1}{2}\frac{k}{\epsilon}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \tag{9}$$

and

$$\omega_{ij} \equiv \frac{1}{2}\frac{k}{\epsilon}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right) \tag{10}$$

The qualitative considerations indicate that the source of dissipation is an increasing function of χ and, tentatively, it will be supposed to be a linear function. Thus, the following modified form of the dissipation equation is proposed

$$\frac{D\epsilon}{Dt} = \frac{\partial}{\partial x_i}\frac{\mu_{\text{eff}}}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial x_i} + \frac{\epsilon^2}{k}\left(C_{\epsilon 1}\frac{P}{\epsilon} - C_{\epsilon 2} + C_{\epsilon 3}\chi\right) \tag{11}$$

where $C_{\epsilon 3}$ is a positive constant to be determined.

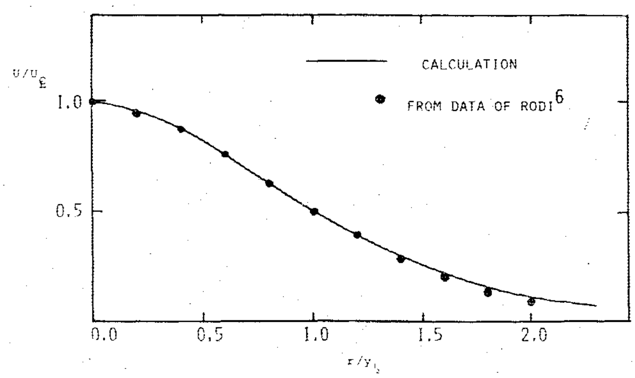

Fig. 1 Axisymmetric jet: mean-velocity profile.

Results and Discussion

The measured spreading rate of the round jet, $dy_{1/2}/dx$, is 0.086 whereas the unmodified k-ϵ model, with the given constants, gives $dy_{1/2}/dx = 0.125$. With the modified version, a value of $C_{\epsilon 3}$ of 0.79 reproduces the measured spreading rate and the velocity profile, Fig. 1, is in excellent agreement with the experimental data.[6] (These calculations were performed numerically using a finite-difference procedure.[7])

These results cannot be claimed as strong supporting evidence for the modified dissipation equation (11) since the constant was chosen to produce the correct spreading rate. It can be claimed, however, that a physically based explanation of the round jet's lower spreading rate has been provided and, from the same reasoning, a modification to the dissipation equation has been suggested; this modification is generally applicable (since it refers only to local quantities in an invariant manner) and results in the accurate calculation of the velocity profiles as well as the correct spreading rates of both round and plane jets.

There is no reason to suppose that the round jet is a singular case. Vortex stretching can be expected to affect more complex flows and, indeed, the modified model provides a quantitative prediction of its influence. For two-dimensional flow fields, however complex, there can be no vortex stretching and so the modification has no effect upon the model; for axisymmetric flows with recirculation, the effect may be considerable and it may provide an explanation, at least in part, of the k-ϵ model's inability to represent such flows accurately.[2] In this instance (axisymmetric flows without swirl) the vortex-stretching invariant χ becomes,

$$\chi = \frac{1}{4}\left(\frac{k}{\epsilon}\right)^3\left(\frac{\partial U}{\partial r} - \frac{\partial V}{\partial x}\right)^2\frac{V}{r} \tag{12}$$

The calculations of these flows represents the best further test of the model but several complicating factors[2,8] make this a not-inconsiderable task.

Three-dimensional flows are also subject to the effect of vortex stretching and, again, the model may provide more realistic representations of these flows. However, the difficulties both of measuring and of calculating three-dimensional flows make them a less useful test.

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Natural Frequencies of a Cantilever with an Asymmetrically Attached Tip Mass

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BHAT and co-workers[1,2] have derived the exact frequency equation for a uniform cantilever beam carrying a tip mass that is slender in the axial direction. The centroid of the tip mass was assumed to lie on the extended neutral axis of the beam when it is in its undeformed configuration. The significant distinct feature of those investigations was that the centroid of the tip mass did not coincide with its point of attachment to the beam. In view of the importance of this problem in airplane and missile design, it is of interest to consider a closely related problem, namely, that in which the centroid of the tip mass does not coincide with its point of attachment to the beam and in which it is also offset an arbitrary distance perpendicular to the extended neutral axis of the beam. In this case, the longitudinal and transverse deflections in the beam become coupled through the boundary conditions because of the presence of the asymmetrically oriented tip mass (see Fig. 1).

Consider a slender elastic cantilever of length ℓ and density ρ carrying a rigid tip mass m that is undergoing longitudinal and flexural deflections $u(x_1,t)$ and $w(x_1,t)$, respectively. Following a derivation similar to that presented in Ref. 3, one can show that the equations of motion for small deflections are

$$\left.\begin{aligned} EAu,_{11} &= \rho A\ddot{u}, \\ EIw,_{1111} + \rho A\ddot{w} &= 0, \end{aligned}\right\} 0<x_1<\ell,\ 0<t \qquad (1)$$

where $u,_1 = \partial u/\partial x_1$, $\dot{u} = \partial u/\partial t$, etc., E denotes Young's modulus for the beam, A its constant cross-sectional area, and I its moment of inertia. The boundary conditions are

$$u = w = w,_1 = 0 \qquad (2)$$

at the clamped end $x_1 = 0$ and

$$EAu,_1 + m\ddot{u} - mr_1\ddot{w},_1 = 0 \qquad (3)$$

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$$EIw,_{11} + (mc^2 + J)\ddot{w},_1 + mc\ddot{w} - mr_1\ddot{u} = 0 \qquad (4)$$

$$EIw,_{111} - m\ddot{w} - mc\ddot{w},_1 = 0 \qquad (5)$$

at the free end $x_1 = \ell$, where r_1 is the distance from o to C (see Fig. 1), C being the centroid of the tip mass, c is the half thickness of the tip mass (assumed here to be symmetric about the vertical axis through o and C), and J is its moment of inertia, with $J = ma_0^2$, a_0 being its radius of gyration.

For free harmonic oscillation of dimensionless natural frequency ω, one may assume that

$$u(x_1,t) = u(x)\cos\omega\tau, \quad w(x_1,t) = w(x)\cos\omega\tau$$

so that Eqs. (1-5) can be expressed in dimensionless form as

$$u''(x) + \omega^2 u(x) = 0, \quad w^{IV}(x) - \lambda^4 w(x) = 0, \quad 0<x<1 \qquad (6)$$

$$u(0) = w(0) = w'(0) = 0 \qquad (7)$$

$$u'(1) - \mu\omega^2 u(1) + \mu r\omega^2 w'(1) = 0 \qquad (8)$$

$$w''(1) - \mu(\xi^2 + a^2)\lambda^4 w'(1) - \mu\xi\lambda^4 w(1) + \mu r\lambda^4 u(1) = 0 \qquad (9)$$

$$w'''(1) + \mu\lambda^4 w(1) + \mu\xi\lambda^4 w'(1) = 0 \qquad (10)$$

where $u' = du/dx$, etc., and where the following variables and dimensionless parameters have been introduced:

$$x = x_1/\ell, \quad \tau = t/\ell(\rho/E)^{1/2}, \quad r = r_1/\ell, \quad \xi = c/\ell$$

$$\mu = m/\rho A\ell, \quad \alpha^2 = I/A\ell^2, \quad \lambda^2 = \omega/\alpha, \quad a = a_0/\ell \qquad (11)$$

Solving the eigenvalue problem in Eqs. (6-10), one obtains the following frequency equation:

$$\begin{aligned} &(\cos\omega - \mu\omega\sin\omega)\{1 + (\mu a\lambda^2)^2 + [1 - (\mu a\lambda^2)]\cos\lambda\cosh\lambda \\ &+ \mu\lambda[1 - \lambda^2(\xi^2 + a^2)]\cos\lambda\sinh\lambda - \mu\lambda[1 + \lambda^2(\xi^2 + a^2)] \\ &\times \sin\lambda\cosh\lambda - 2\mu\xi\lambda^2\sin\lambda\sinh\lambda\} + (\mu r)^2\omega\lambda^3\sin\omega[\mu\lambda(1 \\ &- \cos\lambda\cosh\lambda) - \cos\lambda\sinh\lambda - \sin\lambda\cosh\lambda] = 0 \end{aligned} \qquad (12)$$

In the event that the dimensionless coupling parameter r vanishes, Eq. (12) reduces to two separate frequency equations, namely, that reported in Refs. 1 and 2 for the free flexural vibrations of a cantilever with a slender tip mass and that for the free extension vibrations of a fixed-free bar carrying a tip mass at its free end, i.e., $\cot\omega = \mu\omega$.

For the purpose of determining numerically the natural frequencies ω_n $(n = 1,2,3,\ldots)$ from Eq. (12), it is necessary to specify the geometry of the beam and the tip mass. Suppose, for the sake of example, that the beam has a rectangular cross section of dimensions $2b_1 \cdot h$, so that $A = 2b_1 h$ and $I = 2hb_1^3/3$. Let the tip mass be a rectangular parallelopiped of length ℓ_0, thickness $2c$, and depth h with mass $m = 2\rho_0 ch\ell_0$, where ρ_0 is the density of the tip mass material. Moreover, consider the special case of $|oP| = b_1$ (see Fig. 1). Then

$$\alpha = \frac{b_1}{\sqrt{3}}, \quad \mu = \frac{\rho_0 c\ell_0}{\rho b_1 \ell}, \quad r = \frac{(\ell_0 - 2b_1)}{2\ell}$$

$$a = \frac{1}{\sqrt{3}}(c^2 + \ell_0^2 - 3b_1\ell_0 + 3b_1^2)$$

For specific numerical values of some of the parameters, suppose that $\rho_0/\rho = 2.945$, $b_1 = 0.1$ cm, $c = 0.2$ cm, and $\ell = 5$ cm, while the value of ℓ_0 shall be varied, which, then, implies that the tip mass, the centroid, and the radius of inertia parameters μ, r, and a, respectively, vary as well. In this case, $\alpha = 0.01155$.